

# Speed, Accuracy and Caution: the Timing of Choices Under Risk Aversion

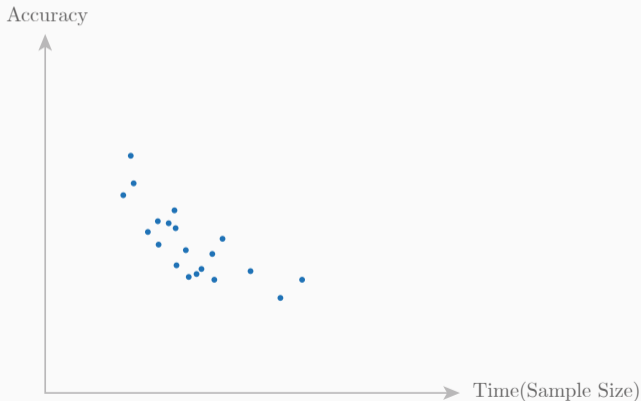
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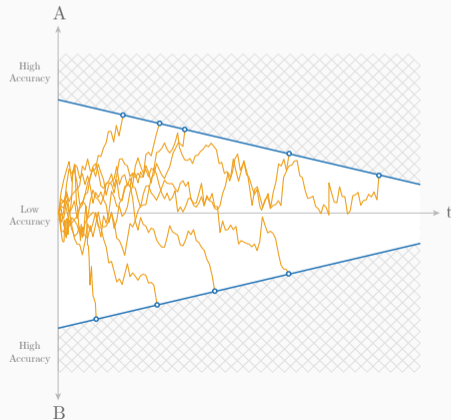
## Motivation

- Large body of literature finds a negative correlation between speed and accuracy, [Fried and Peterson \(1969\)](#), [Swenson \(1972\)](#), [Luce et al. \(1986\)](#), [Ratcliff and Smith \(2004\)](#), [Ratcliff and McKoon \(2008\)](#), [Brown et al. \(2011\)](#), [Reshidi et al. \(2022\)](#).



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- Which can be explained by decreasing optimal-stopping thresholds in time. This has led to multiple alternative-specifications or behavioral explanations. To name a few:
  - Bias against “throwing good money after bad” [Fried and Peterson \(1969\)](#).
  - ...
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  - Unknown payment of state A and B [Fudenberg et al. \(2018\)](#).
- We ask: **Could risk-aversion alone lead to time dependent stopping rules?**

- Analyze the richness of optimal boundaries under risk averse utility functions.
- Provide a method which is informative of the optimal boundaries.

## Static Problem

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There is an underlying state  $\omega \in \{A, B\}$ . Unobserved by the agent; agent has a prior  $P(\omega = A) = p_0$ .

Agent chooses among two alternatives  $a$  or  $b$ .

Receives a bonus  $x$  if choice matches state, no bonus otherwise.

Agent has initial wealth  $w$ .

Agent chooses how much information to acquire.

Information:

$$S_t = \begin{cases} \tilde{\mu}t + \rho W_t & \text{if } \omega = A \\ -\tilde{\mu}t + \rho W_t & \text{if } \omega = B \end{cases}$$

Timing: Choose  $t \rightarrow$  given information choose  $a$  or  $b$ .

Agent pays cost  $c \cdot t$  for information collection.

$$u(t, a|\omega = A) = u(w + x - ct) \quad u(t, a|\omega = B) = u(w - ct)$$

$u(\cdot)$  - some concave utility function.



Define  $\mu = \frac{2\tilde{\mu}^2}{\rho^2}$ . Set  $p_0 = 1/2$  for ease of exposition.

$$\begin{aligned} & \max_t \mathbb{E} \left[ \frac{1}{2} \left( \mathbb{P} \left( \frac{e^{S_t(\mu)}}{1 + e^{S_t(\mu)}} \geq \frac{1}{1 + e^{S_t(\mu)}} \right) u(w + x - ct) + \mathbb{P} \left( \frac{e^{S_t(\mu)}}{1 + e^{S_t(\mu)}} < \frac{1}{1 + e^{S_t(\mu)}} \right) u(w - ct) \right) \right. \\ & \left. + \frac{1}{2} \left( \mathbb{P} \left( \frac{e^{S_t(-\mu)}}{1 + e^{S_t(-\mu)}} < \frac{1}{1 + e^{S_t(-\mu)}} \right) u(w + x - ct) + \mathbb{P} \left( \frac{e^{S_t(-\mu)}}{1 + e^{S_t(-\mu)}} \geq \frac{1}{1 + e^{S_t(-\mu)}} \right) u(w - ct) \right) \right] \end{aligned}$$

$$\max_t p(t) u(w + x - ct) + (1 - p(t)) u(w - ct)$$

$$p(t) = \frac{1}{2} \left( \operatorname{erf} \left( \frac{\sqrt{\mu t}}{2} \right) + 1 \right)$$

f.o.c

$$\frac{u(w + x - ct) - u(w - ct)}{p(t)u'(w + x - ct) + (1 - p(t))u'(w - ct)} p'(t) = c$$

### Proposition

$t^*$  is independent of  $w \iff u$  is CARA.

$$u(\cdot) = \frac{1 - e^{-\alpha(\cdot)}}{\alpha}$$

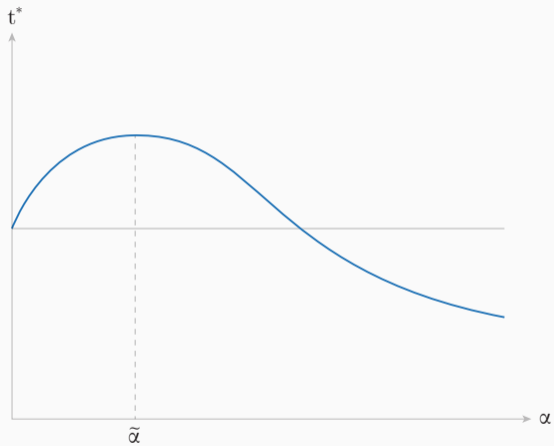
$$\frac{\int_{w-ct}^{w+x-ct} u'(s) ds}{p(t)u'(w+x-ct) + (1-p(t))u'(w-ct)} p'(t) = \frac{e^{2\alpha x} - 1}{2\alpha(p(t) - (p(t) - 1)e^{2\alpha x})} p'(t) = c$$

$t^*(\alpha)$  is implicitly defined by

$$\frac{\mu e^{-\frac{\mu t^*}{4}}}{4\sqrt{\pi\mu t^*} \left( \frac{1}{2}\alpha \operatorname{erfc}\left(\frac{\sqrt{\mu t^*}}{2}\right) + \frac{\alpha}{e^{\alpha x} - 1} \right)} = c$$

$t^*$  is a function of only  $\alpha, x, c$  and  $\mu$ . From the implicit function theorem  $\exists \tilde{\alpha} > 0$  such that

$$\frac{\partial t^*}{\partial \alpha} > 0 \quad \text{if } \alpha < \tilde{\alpha} \qquad \frac{\partial t^*}{\partial \alpha} \leq 0 \quad \text{if } \alpha \geq \tilde{\alpha}$$



$$u(\cdot) = \sum_i \gamma_i \text{CARA}_{\alpha_i}(\cdot) \quad \gamma_i \geq 0 \quad \sum_i \gamma_i = 1$$

Without loss assume  $\alpha_i \leq \alpha_{i+1}$ . Let  $t_i^*$  be the arg max under  $\text{CARA}_{\alpha_i}(\cdot)$ .

### Proposition

$$\forall w \quad \min_i t_i^* \leq t^*(w) \leq \max_i t_i^*$$

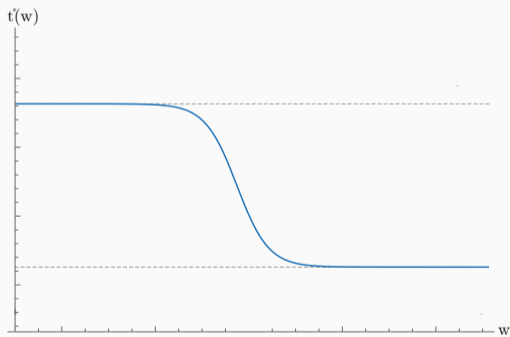
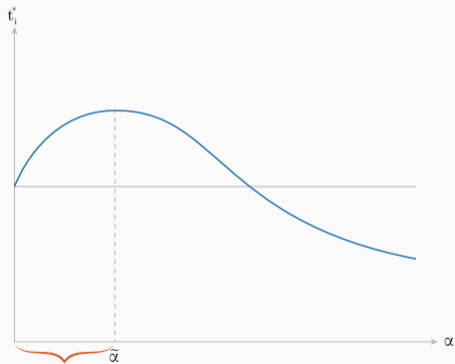
$$\lim_{w \rightarrow \infty} t^*(w) = t_1^*$$

$$\lim_{w \rightarrow -\infty} t^*(w) = t_j^*$$

$$t^*(w) \begin{cases} \text{may be quasiconcave} & \text{if } \min_i \alpha_i \leq \tilde{\alpha} \leq \max_i \alpha_i \\ \text{monotonically converges} & \text{otherwise} \end{cases}$$

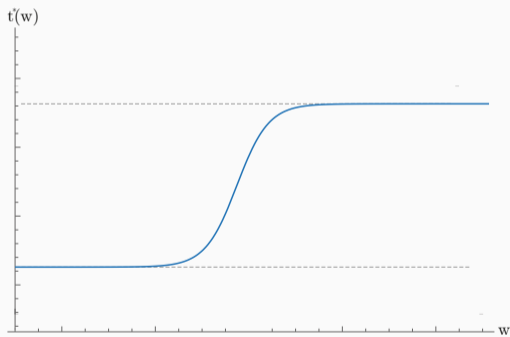
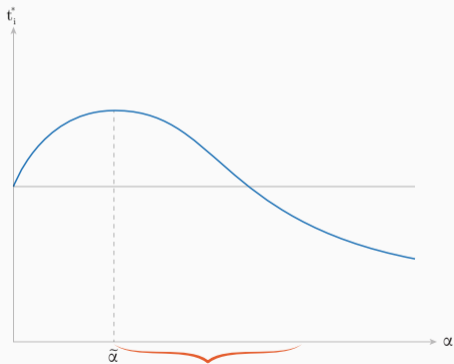
$$u(\cdot) = \sum_i \gamma_i \text{CARA}_{\alpha_i}(\cdot)$$

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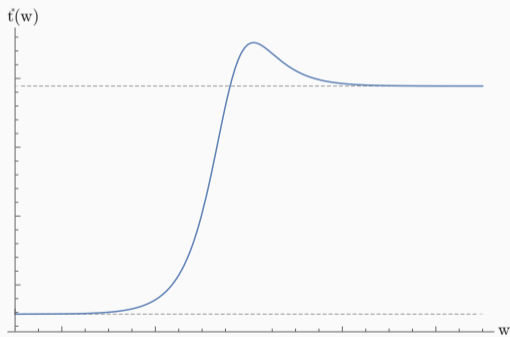
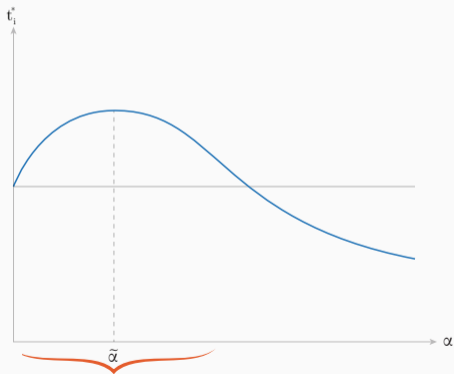
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## Dynamic Problem

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There is an underlying state  $\omega \in \{A, B\}$ . Unobserved by the agent; agent has a prior.  
Agent chooses among two alternatives  $a$  or  $b$ .  
Receives a bonus  $x$  if choice matches state, no bonus otherwise.  
Agent has initial wealth  $w$ .

Information arrives continuously as long as an alternative has not been chosen.  
Information: Wiener process  $dS$  with drift  $\tilde{\mu}(-\tilde{\mu})$  and variance  $\rho^2$  if the state is  $A(B)$ .  
Agent pays flow cost  $c$  for information collection.

$$u(a, t|\omega = A) = u(w + x - ct) \quad u(a|\omega = B) = u(w - ct)$$

$$u(\cdot) = \text{CARA}_\alpha(\cdot)$$

Ito's lemma leads to

$$\frac{\partial V}{\partial t} = \frac{\partial V}{\partial \theta} \mu \frac{e^\theta - 1}{e^\theta + 1} + \mu \frac{\partial V}{\partial \theta^2}.$$

Value matching (boundary condition)

$$V(z^*(t), t) = p(z^*(t)) \frac{1 - e^{-\alpha(w+x-ct)}}{\alpha} + (1 - p(z^*(t))) \frac{1 - e^{-\alpha(w-ct)}}{\alpha}$$

+Smooth pasting condition

+Initial value condition

$$V(\theta, t|z) = \frac{1}{\alpha} - \frac{e^{\alpha ct} \left( \operatorname{sech} \left( \frac{\theta}{2} \right) e^{-\frac{1}{2}\alpha(2w+x)} \cosh \left( \frac{1}{2}(z - \alpha x) \right) \cosh \left( \frac{1}{2}\theta \sqrt{1 - \frac{4\alpha c}{\mu}} \right) \operatorname{sech} \left( \frac{1}{2}z \sqrt{1 - \frac{4\alpha c}{\mu}} \right) \right)}{\alpha}$$

Let  $z^*(\alpha)$  be the value that sets  $\frac{\partial V(z|\theta)}{\partial z} = 0$ .

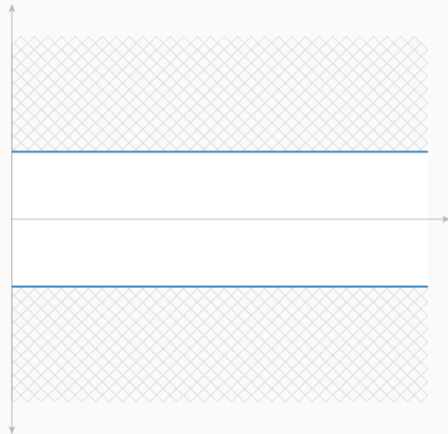
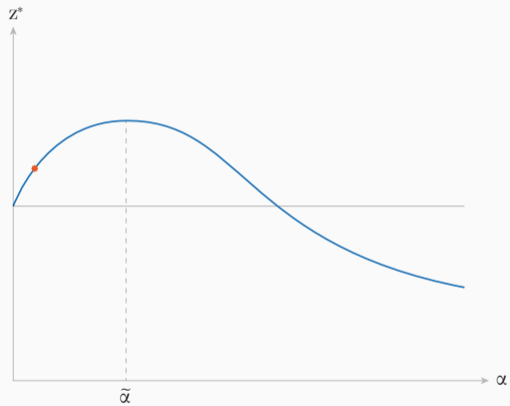
$$\sinh \left( \frac{1}{2} (z^* - \alpha x) \right) - \sqrt{1 - \frac{4\alpha c}{\mu}} \cosh \left( \frac{1}{2} (z^* - \alpha x) \right) \tanh \left( \frac{1}{2} z^* \sqrt{1 - \frac{4\alpha c}{\mu}} \right) = 0$$

$z^*$  is a function of only  $\alpha, x, c$  and  $\mu$ .

From the implicit function theorem

$$\frac{\partial z^*}{\partial \alpha} > 0 \quad \text{if } \alpha < \tilde{\alpha} \qquad \frac{\partial z^*}{\partial \alpha} \leq 0 \quad \text{if } \alpha \geq \tilde{\alpha}$$

# CARA Optimal Boundary



$$u(\cdot) = \sum_i \gamma_i \text{CARA}_{\alpha_i}(\cdot) \quad \gamma_i \geq 0 \quad \sum_i \gamma_i = 1$$

Without loss assume  $\alpha_i \leq \alpha_{i+1}$ . Let  $V_i(z, t)$  be the value function for  $\alpha_i$

$$\frac{\partial V}{\partial t} = \frac{\partial V}{\partial \theta} \mu \frac{e^\theta - 1}{e^\theta + 1} + \mu \frac{\partial V}{\partial \theta^2}$$

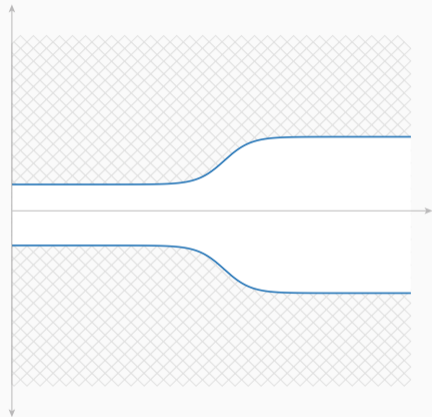
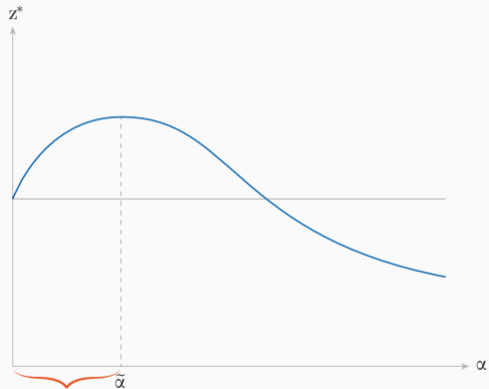
Value matching (boundary condition)

$$V(z^*(t), t) = p(z^*(t)) \sum_{i=1}^n \gamma_i \frac{1 - e^{-\alpha_i(w+x-ct)}}{\alpha} + (1 - p(z^*(t))) \sum_{i=1}^n \gamma_i \frac{1 - e^{-\alpha_i(w-ct)}}{\alpha}$$

+Smooth pasting condition

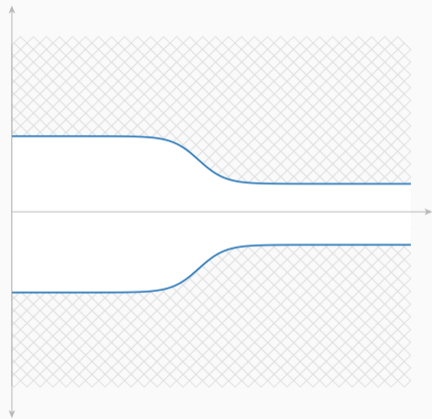
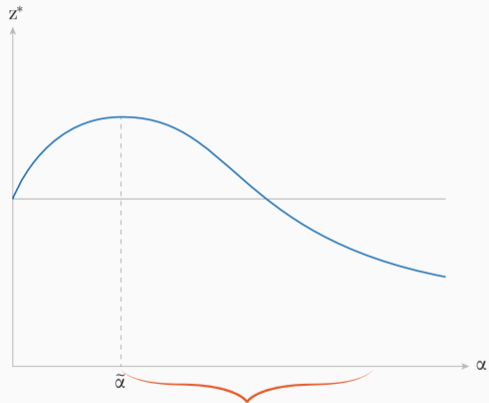
+Initial value condition

# mixed-CARA Optimal Boundary

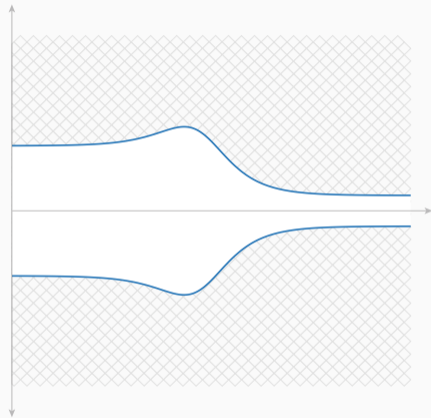
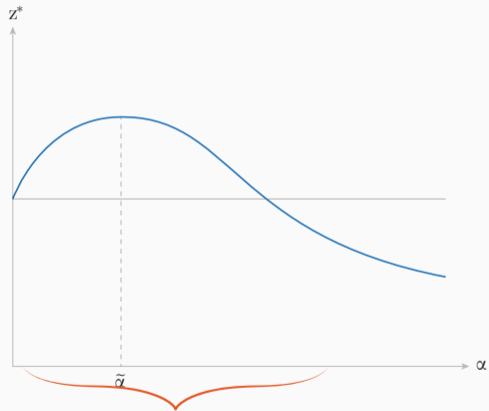




# mixed-CARA Optimal Boundary



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## Proposition

$$\min_i z_i^* \leq z^*(t) \leq \max_i z_i^*$$

$$\lim_{w \rightarrow \infty} z^*(w) = z_1^*$$

$$\lim_{w \rightarrow -\infty} z^*(w) = z_l^*$$

**Paying a fixed price for information of fixed precision** Example, buying a 3\$ news paper.

What if we pull  $ct$  out of  $u(\cdot)$ : curvature for  $w$  and  $x$ , but pay for information in utiles.

Then, let  $\tilde{u} = u(w)$ , re-normalize payments  $\tilde{x} = u(w + x) - u(w)$ .

Win:  $\tilde{u} + \tilde{x}$                       Lose:  $\tilde{u}$

The problem reduces to the linear problem  $\implies$  time-independent boundaries.

From mixed-CARA to any concave  $u(\cdot)$

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- Approximate continuous, concave and increasing functions with linear combinations of CARAs.

## Approximating Concave-Increasing functions

Let  $X \subset \mathbb{R}$  be a compact set, and  $\mathcal{F}$  be the space of all continuous, increasing and concave functions  $f : X \rightarrow \mathbb{R}$ . Define:

$$C = \left\{ f \in \mathcal{F} : f(x) = - \sum_{i=1}^N \beta_i e^{-\alpha_i x}, (\alpha_i, \beta_i) \in \mathbb{R}_+ \times \mathbb{R} \right\}.$$

### Apply Stone-Weierstrass

#### *Lemma*

*C is dense in  $\mathcal{F}$  in the uniform norm.*

Proof: **An application of Stone-Weierstrass.**

- Approximate continuous, concave and increasing functions with linear combinations of CARAs. ✓
- Prove value functions can also be approximated (operator continuity – Shiryaev).



## Approximating Value Functions.

Following Shiryayev (2008), define:

$$Tu(\theta, t) = \mathbb{E}_\pi u(\theta_1, t + 1) \quad Qu(\theta, t) = \max \{u(\theta, t), Tu(\theta, t)\} \quad V_u(\theta, t) = \lim_{n \rightarrow \infty} Q^n u(\theta, t).$$

### *Lemma*

*For any compact set  $X$  and  $\varepsilon > 0$ , let  $\|u - w\|_X < \varepsilon$ , then,  $\|V_u - V_w\|_X < \varepsilon$ .*

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- Prove value functions can also be approximated (operator continuity – Shiryaev). ✓
- Show compactness is not a restriction (Shiryaev).

- Approximate continuous, concave and increasing functions with linear combinations of CARAs. ✓
- Prove value functions can also be approximated (operator continuity – Shiryaev). ✓
- Show compactness is not a restriction (Shiryaev). ✓
- Derive properties of boundaries for mixes of linear combinations of CARAs.

- Concave  $u \implies$  rich set of boundaries.
- Structure for mixed-CARA.
- Soon: structure for any  $u$ .

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